Lecture 9

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Third Edition

MECHANICS OF MATERIALS

CHAPTER

3

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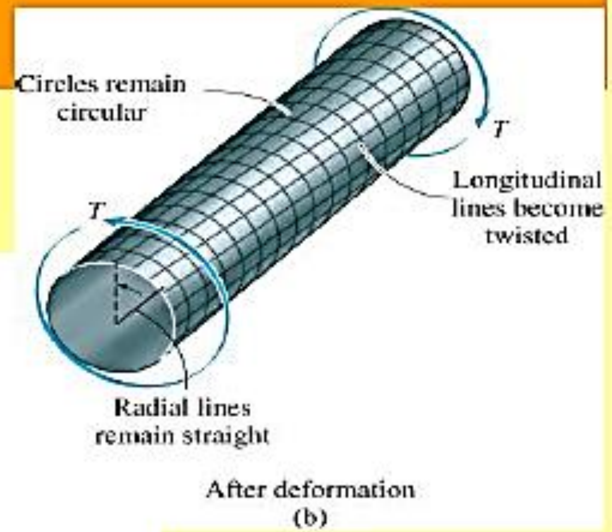
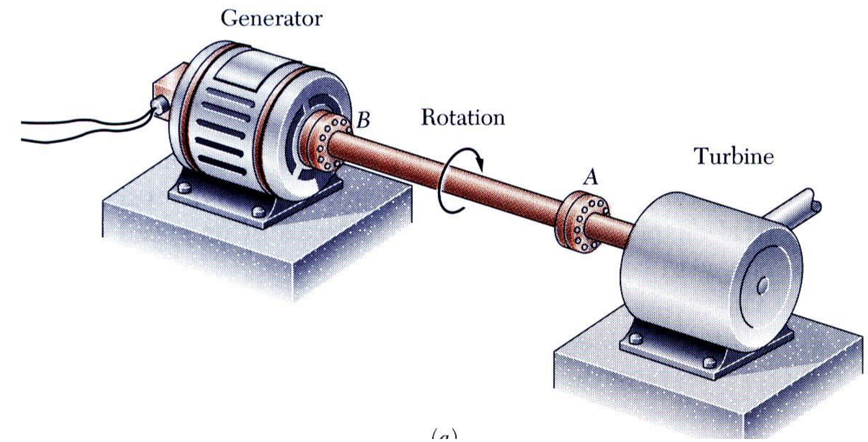
Torsion

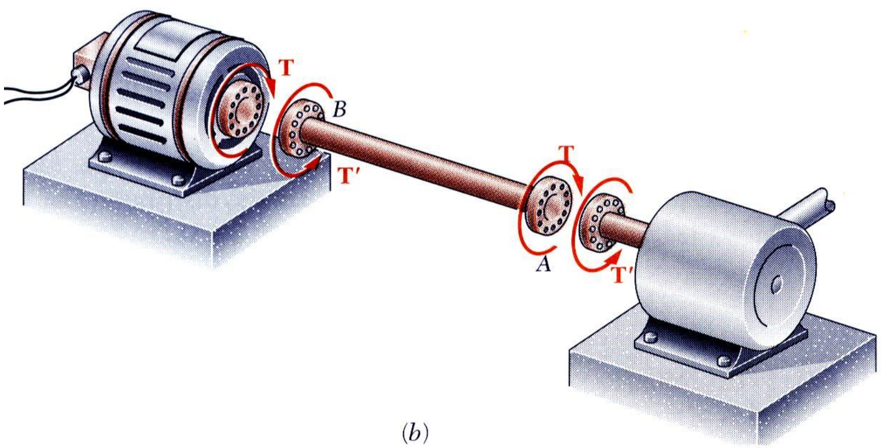
OUTLINE



1. Torsional Deformation of a Circular Shaft
2. The Torsion Formula
3. Twisting Angle
4. Power Transmission
5. Statically Indeterminate Torque-Loaded Members
6. Stress Concentration

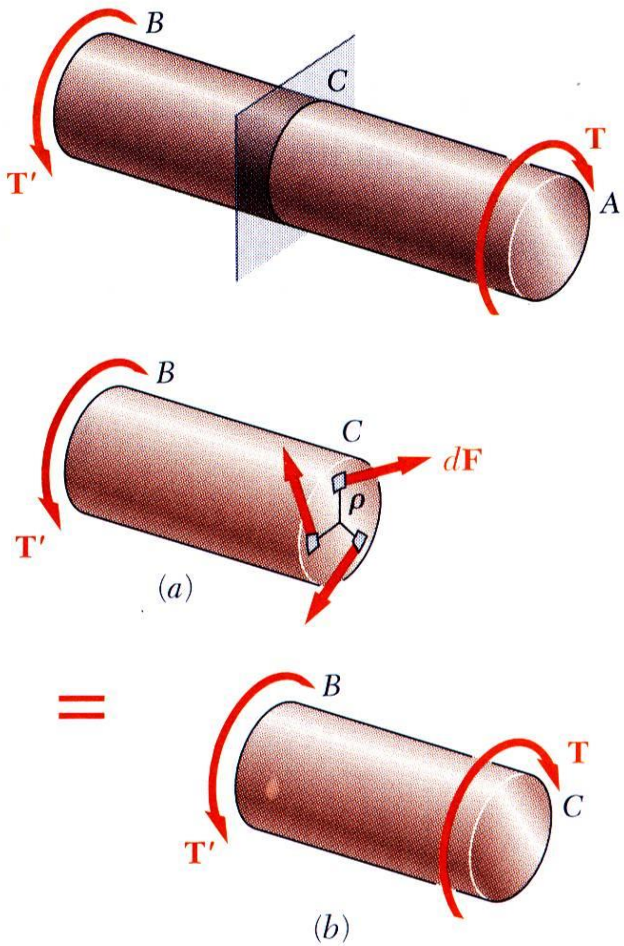
Torsional Loads on Circular Shafts





- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T*’

Net Torque Due to Internal Stresses

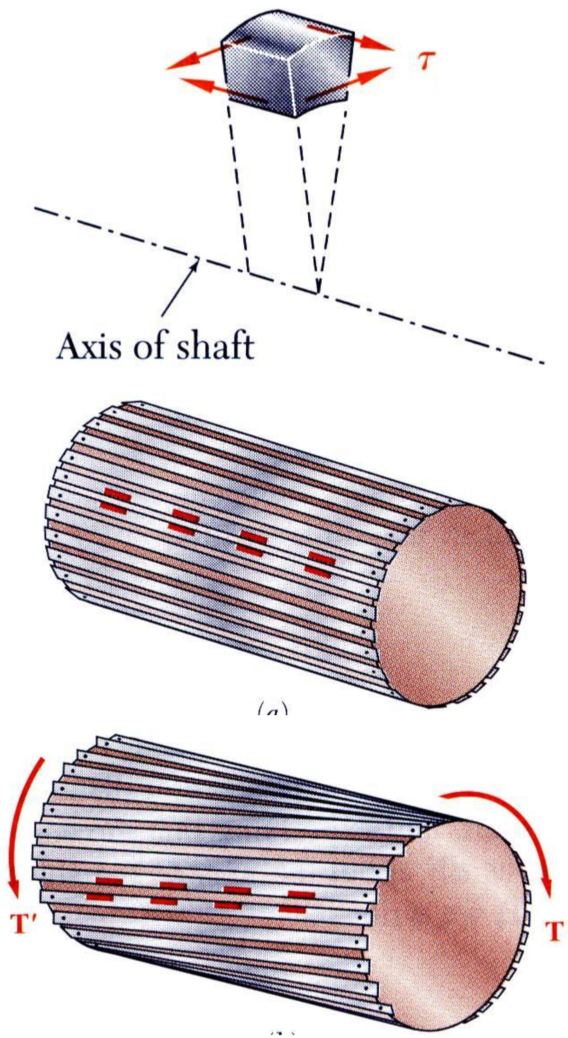


- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

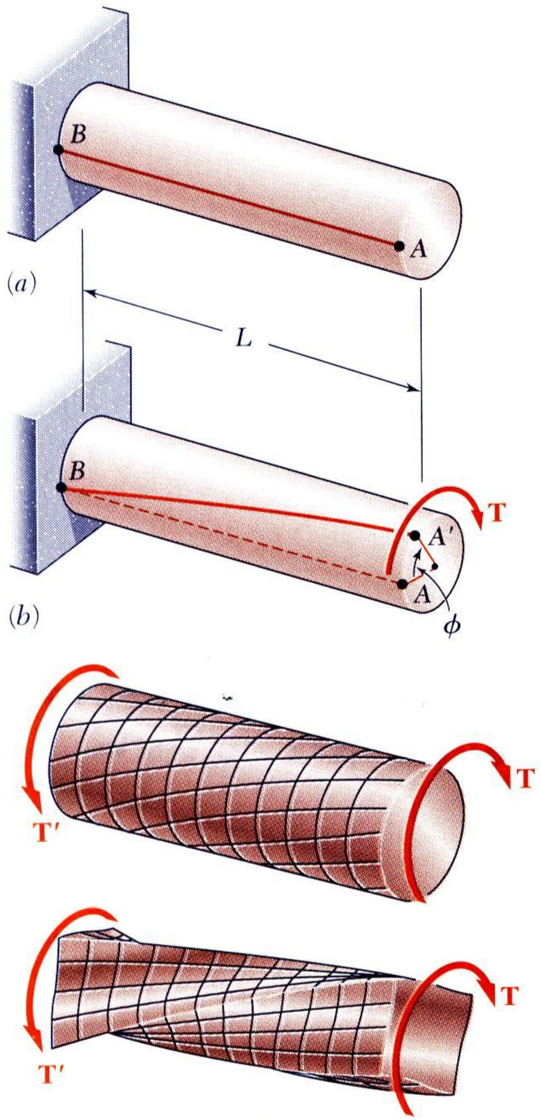
Axial Shear Components



- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

  The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations



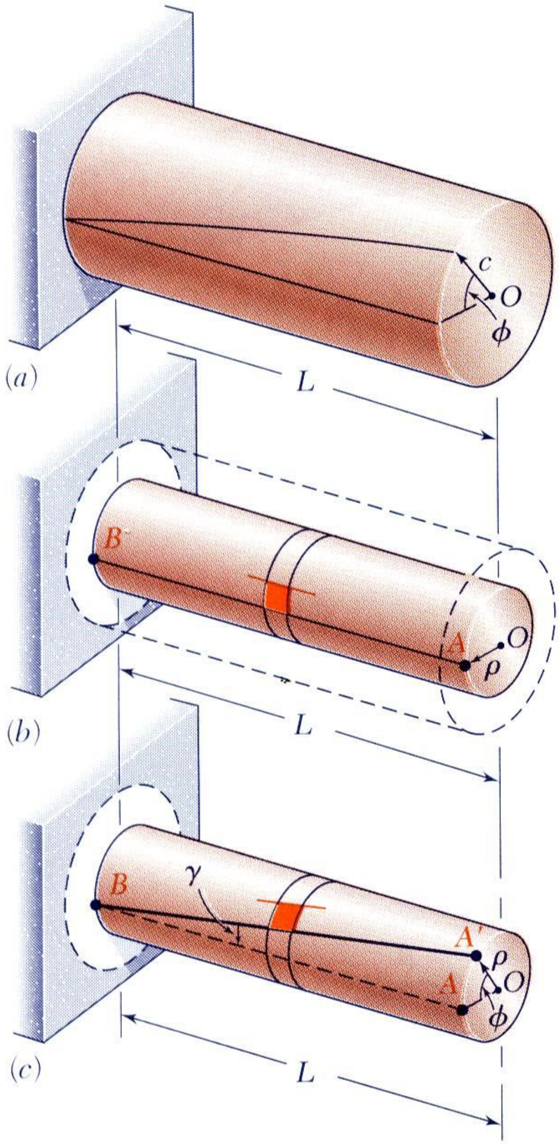
- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Shearing Strain



- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

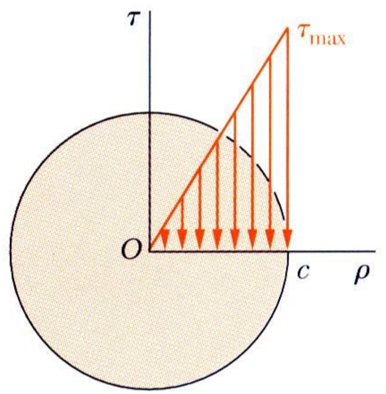
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

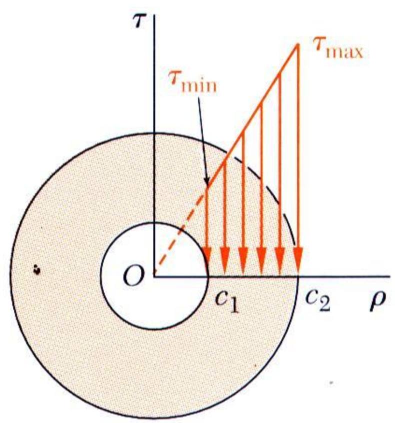
- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range



$$J = \tfrac{1}{2}\pi c^4$$



$$J = \tfrac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

- Multiplying the previous equation by the shear modulus,

  $$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

  From Hooke's Law, $\tau = G\gamma$, so

  $$\tau = \frac{\rho}{c}\tau_{max}$$

  The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

  $$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

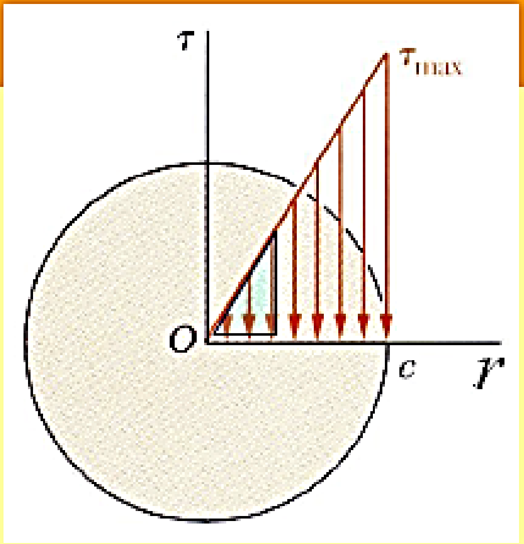
- The results are known as the elastic torsion formulas,

  $$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

$r = \rho$ is the shaft radius

THE TORSION FORMULA

$$\tau_{max} = \frac{Tc}{J}$$



$$J = \frac{1}{2}\pi c^4$$

τ_{max} = max. shear stress in shaft, at the outer surface

T = resultant internal torque acting at x-section, from method of sections & equation of moment equilibrium applied about longitudinal axis

J = polar moment of inertia at x-sectional area

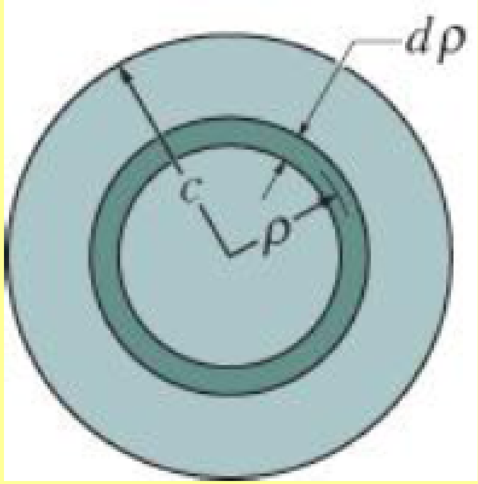
c = outer radius of the shaft

THE TORSION FORMULA

Polar moment of inertia, *J*

for a solid circular shaft, can be determined by:

$$J = \frac{\pi}{32} d^4$$



Note that $r = \rho$ is the shaft radius

- J is a geometric property of the circular area and is always positive. Its common units are:

mm^4 & m^4.

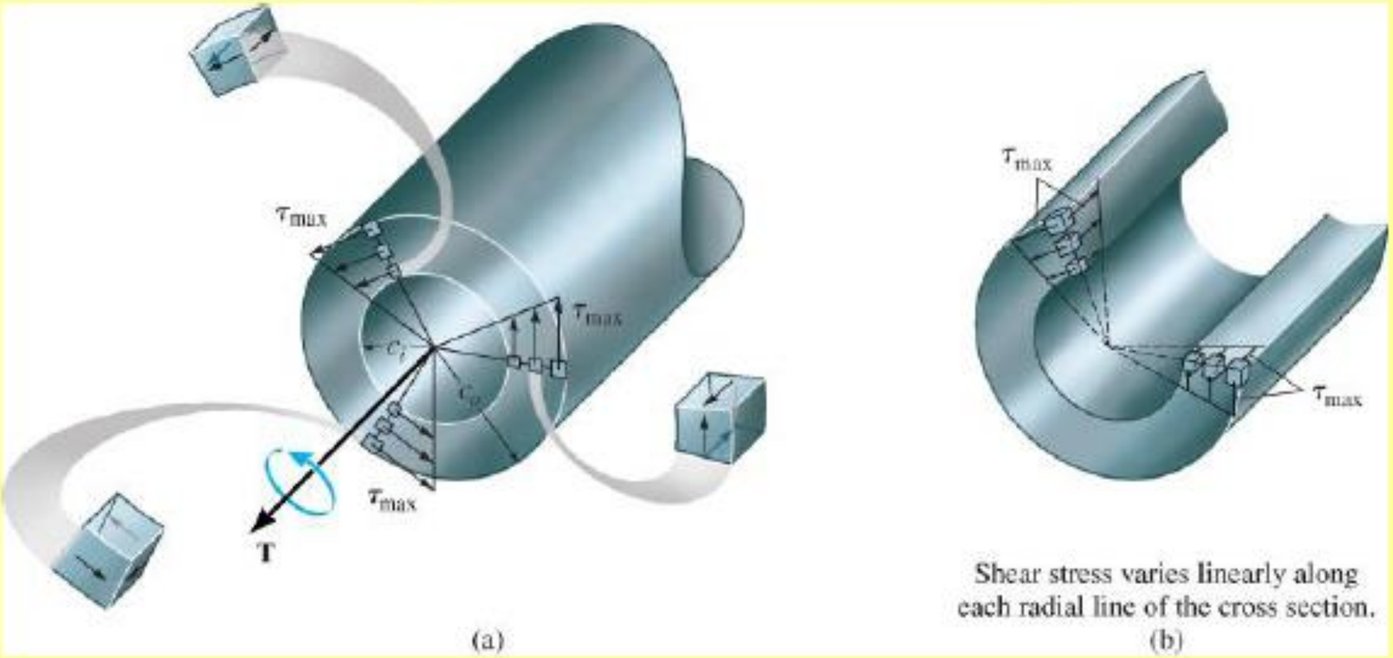
- For a hollow circular shaft,

THE TORSION FORMULA

Polar moment of inertia, *J*

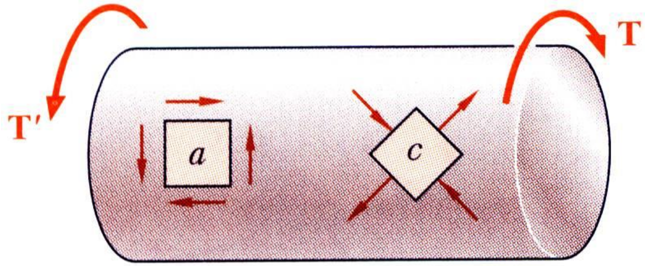
for a tubular circular shaft, can be determined by:

$$J = \frac{\pi}{32}\left(d_o^{\,4} - d_i^{\,4}\right)$$

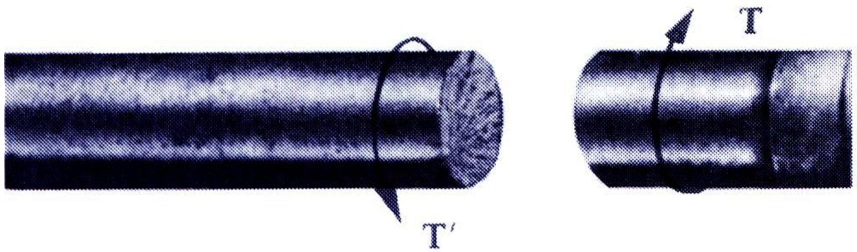


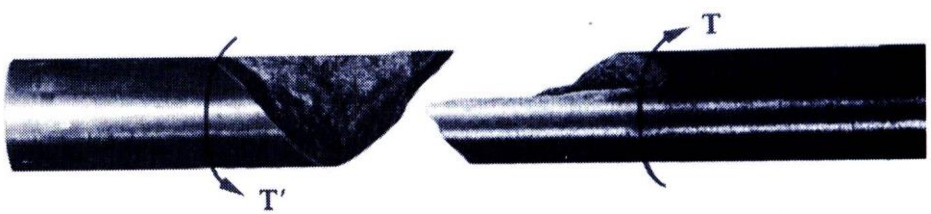
Shear stress varies linearly along each radial line of the cross section.
(b)

Torsional Failure Modes



$$\tau_{max} = \frac{Tc}{J} \qquad \sigma_{45°} = \pm \frac{Tc}{J}$$





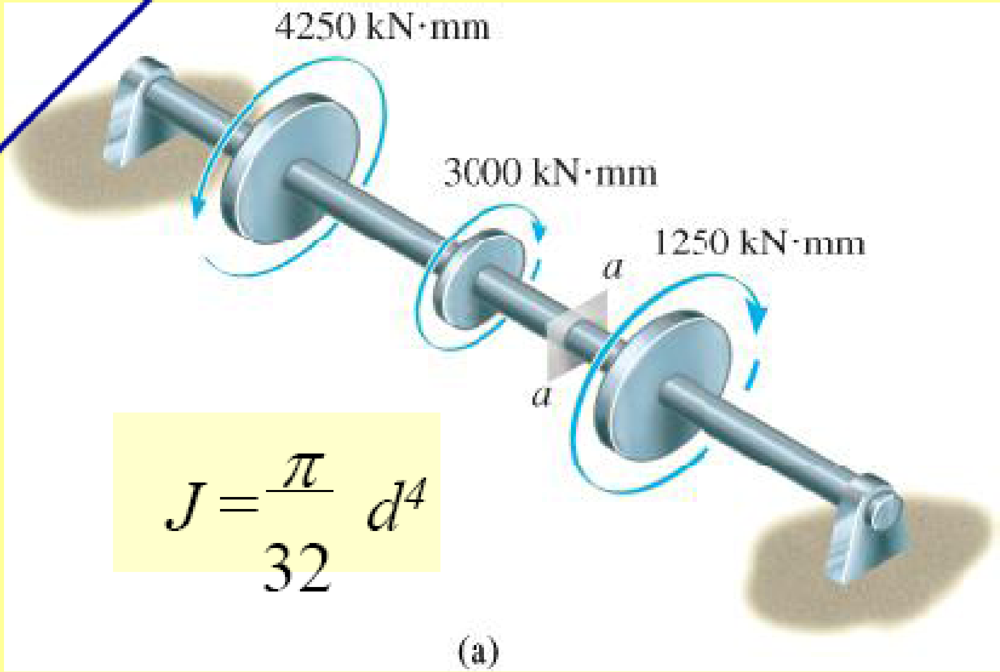
- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45º to the shaft axis.

Example

Shaft shown supported by two bearings and subjected to three torques. **(a) Draw the torque diagram and (b) determine shear stress** developed at **points A and B**, located at section *a-a* of the shaft at 15 and 75 mm radii. $\underline{J = 4.97 \times 10^7\ mm^4}$

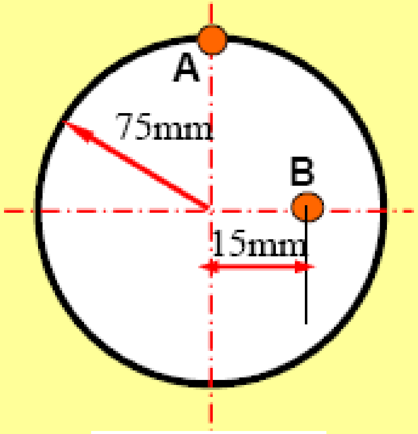
r

c



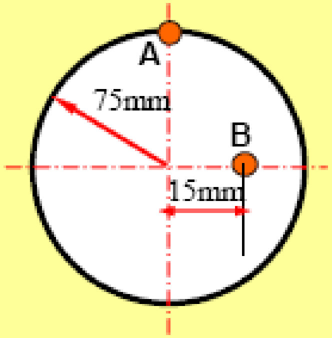
$$J = \frac{\pi}{32} d^4$$

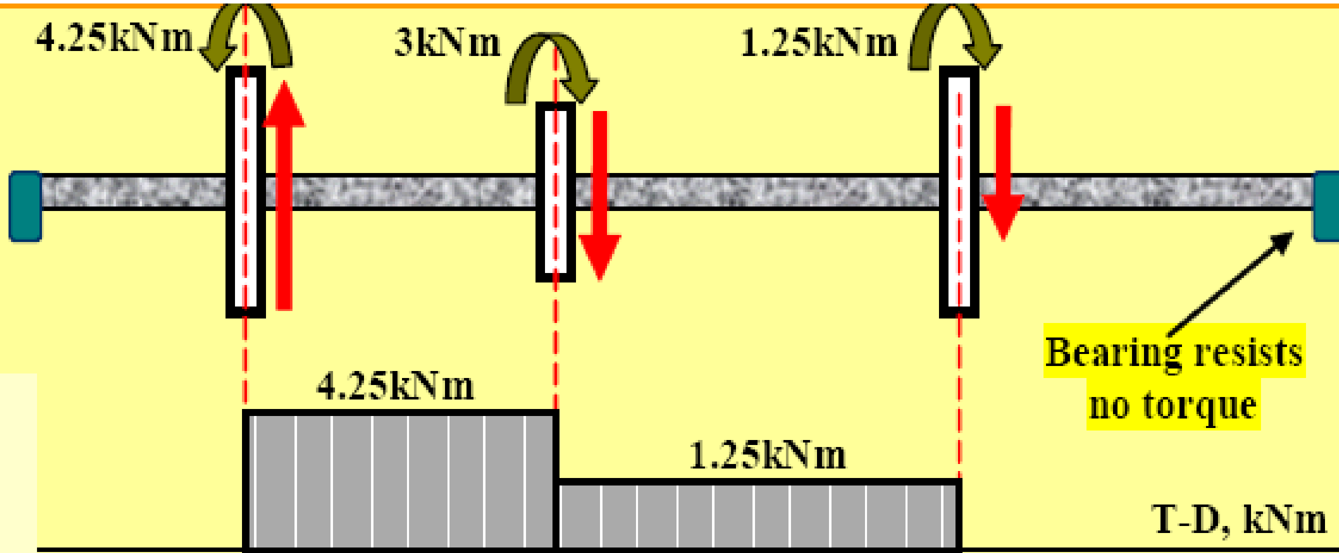
(a)



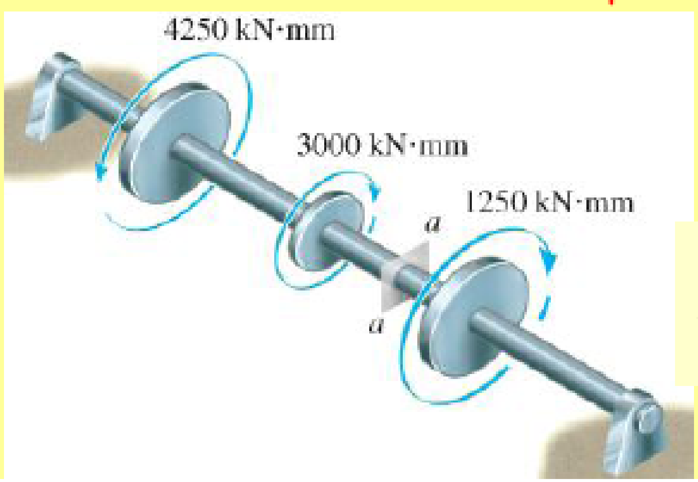
Sec. a-a

Example



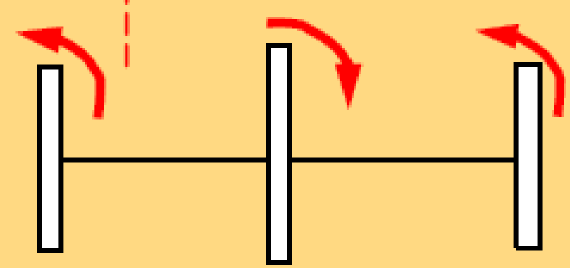


$$\tau = \frac{T\,r}{J}$$



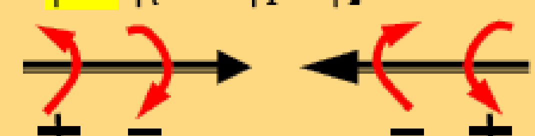
$$J = \frac{\pi}{32} d^4$$

$$J = \frac{\pi}{32} \left(d_o^{\,4} - d_i^{\,4}\right)$$



T-D: *Internal Torque*

ϕ-D: $\phi_t = \pm\,\phi_1 \pm \phi_2$



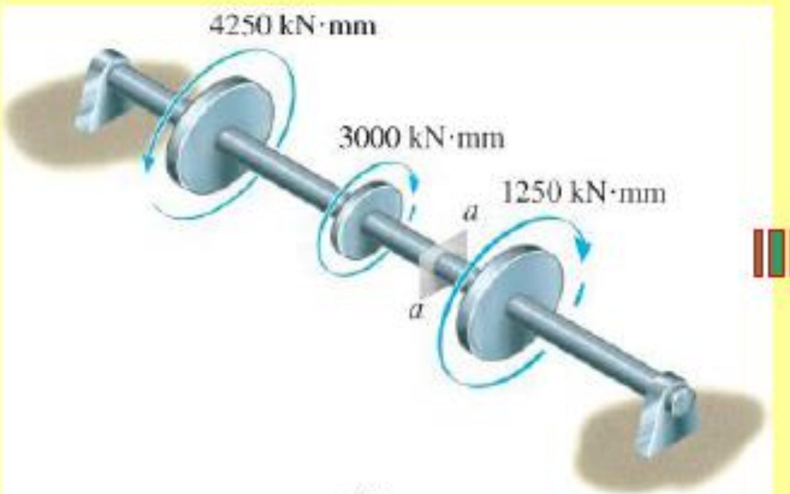
Example

Internal torque

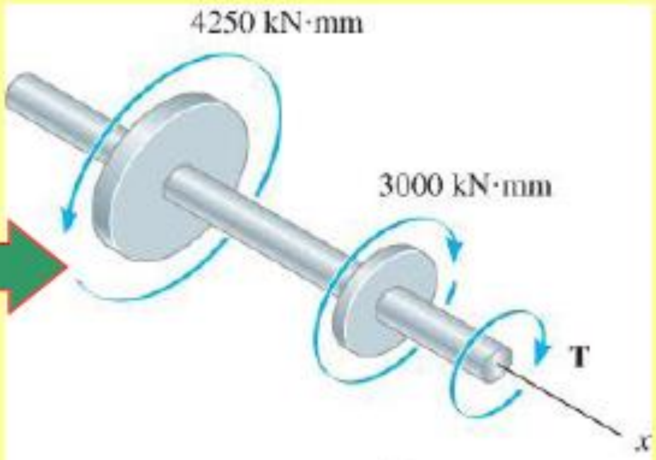
If shaft weight assumed to be negligible, then bearing torque reactions on shaft = 0,

Applied torques satisfy moment equilibrium about shaft's axis.

Internal torque at section *a*-*a* determined from free-body diagram of left segment.



(a)



(b)

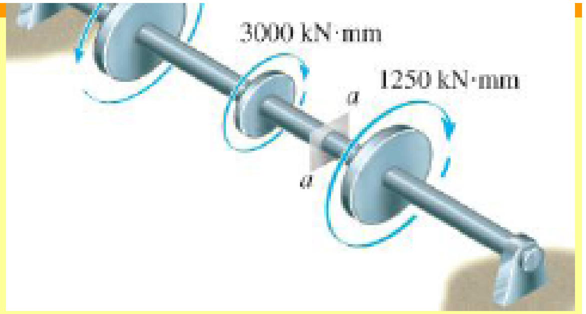
Example

Internal torque:

$\Sigma T x = 0;$

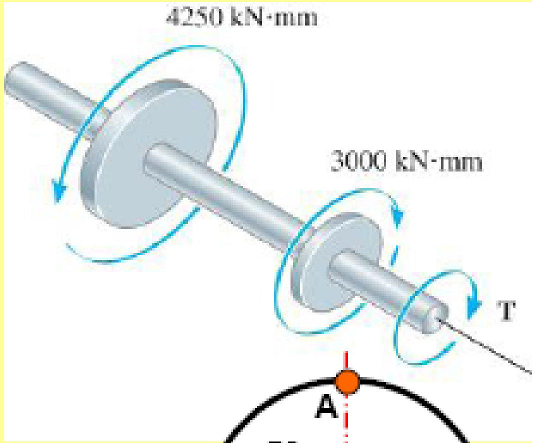
$4250 \text{ kN·mm} - 3000 \text{ kN·mm} - T = 0$

$T = 1250 \text{ kN·mm}$



Section property

$J = \pi(75 \text{ mm})^4 /2 = 4.97 \times 10^7 \text{ mm}^4$



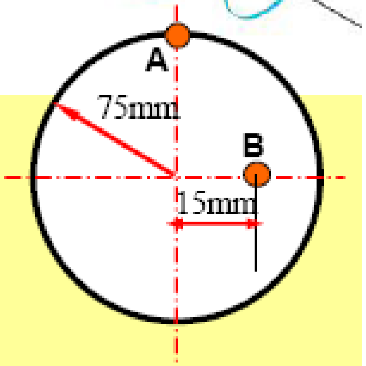
Shear Stresses

Since point A **is at** $r = c = 75$ mm

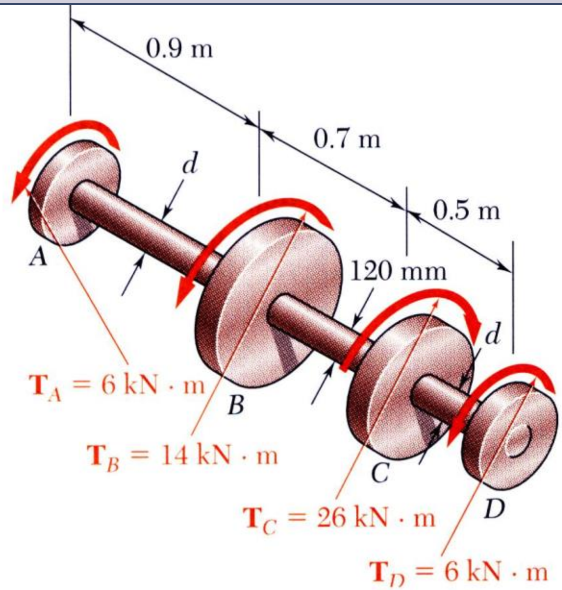
$\underline{\tau_A = Tc/J = \ldots = 1.89 \text{ MPa}}$

Likewise at Pt. B, $r = 15$mm

$\underline{\tau_B = T\,r\,/J = \ldots = 0.377 \text{ MPa}}$



Sample Problem 3.1



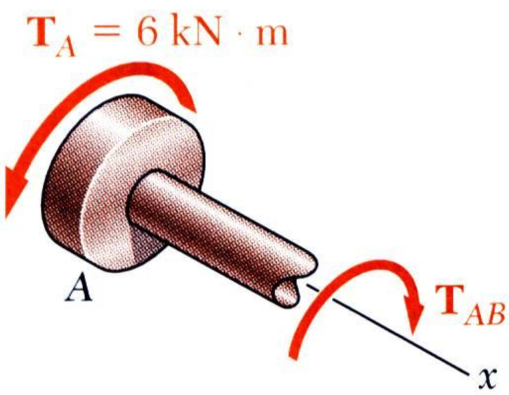
Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

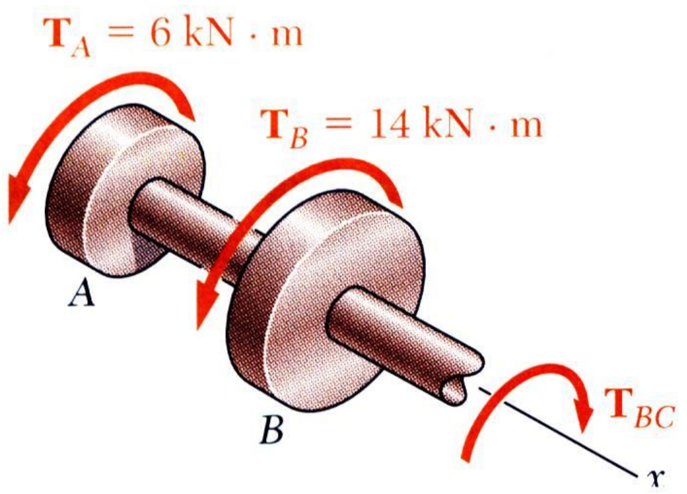
SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

Sample Problem 3.1

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings





$$\sum M_x = 0 = (6\,\mathrm{kN\cdot m}) - T_{AB}$$

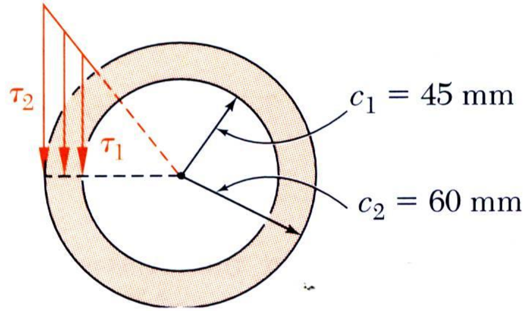
$$T_{AB} = 6\,\mathrm{kN\cdot m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\mathrm{kN\cdot m}) + (14\,\mathrm{kN\cdot m}) - T_{BC}$$

$$T_{BC} = 20\,\mathrm{kN\cdot m}$$

Sample Problem 3.1

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*



$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92 \times 10^{-6}\,\mathrm{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\mathrm{kN\cdot m})(0.060\,\mathrm{m})}{13.92\times 10^{-6}\,\mathrm{m}^4}$$

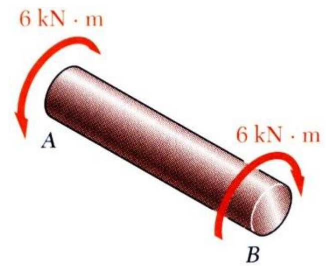
$$= 86.2\,\mathrm{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\mathrm{MPa}} = \frac{45\,\mathrm{mm}}{60\,\mathrm{mm}}$$

$$\tau_{min} = 64.7\,\mathrm{MPa}$$

$$\boxed{\begin{aligned}\tau_{max} &= 86.2\,\mathrm{MPa}\\ \tau_{min} &= 64.7\,\mathrm{MPa}\end{aligned}}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

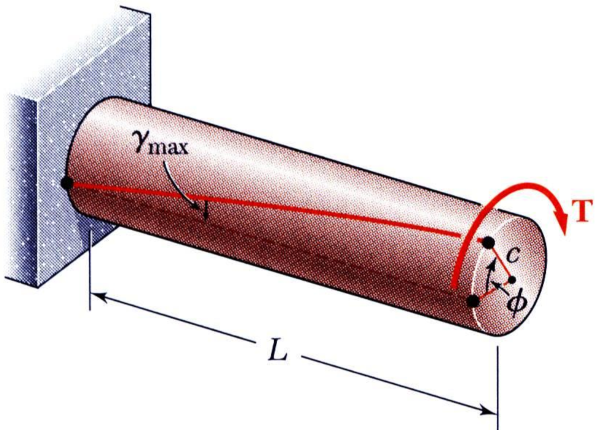


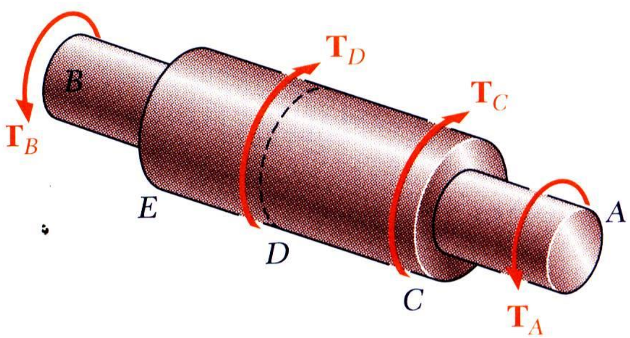
$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\mathrm{kN\cdot m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9 \times 10^{-3}\,\mathrm{m}$$

$$\boxed{d = 2c = 77.8\,\mathrm{mm}}$$

Angle of Twist in Elastic Range





- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

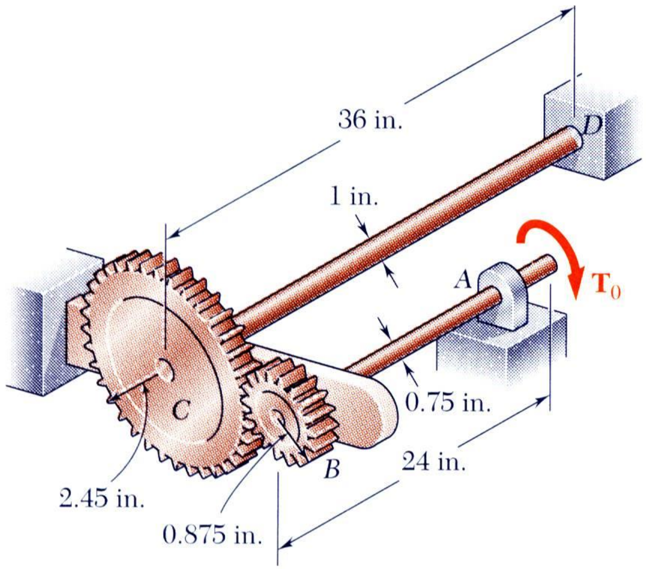
- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft G = 11.2 x 10^6 psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

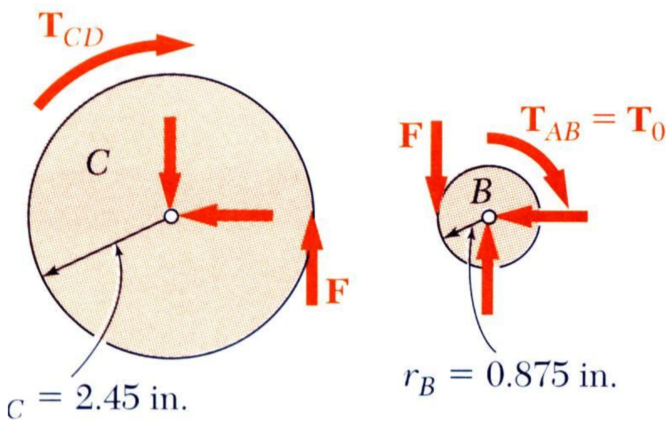
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0
- Apply a kinematic analysis to relate the angular rotations of the gears
- Find the maximum allowable torque on each shaft – choose the smallest
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*

Sample Problem 3.4

SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0

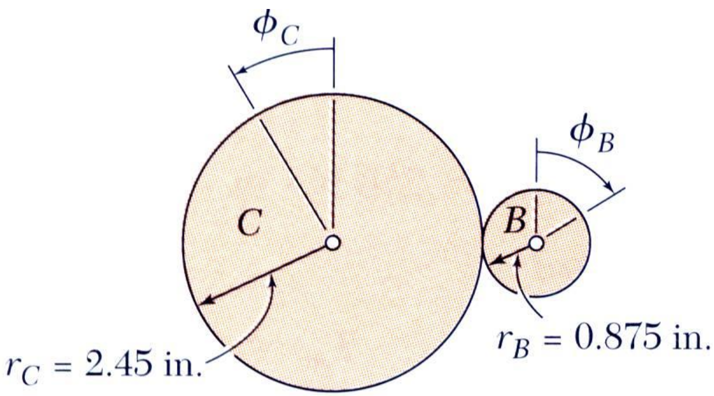


$$\sum M_B = 0 = F(0.875\,\text{in.}) - T_0$$

$$\sum M_C = 0 = F(2.45\,\text{in.}) - T_{CD}$$

$$T_{CD} = 2.8 T_0$$

- Apply a kinematic analysis to relate the angular rotations of the gears



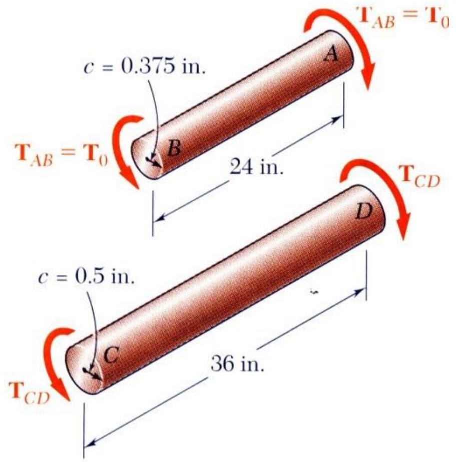
$$r_B \phi_B = r_C \phi_C$$

$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{2.45\,\text{in.}}{0.875\,\text{in.}}\phi_C$$

$$\phi_B = 2.8\phi_C$$

Sample Problem 3.4

- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest



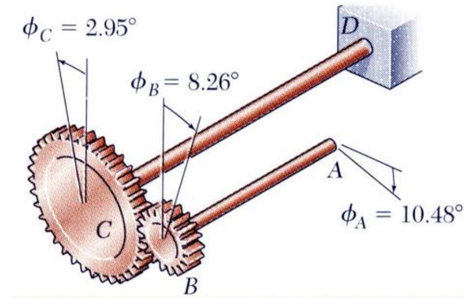
$$\tau_{max} = \frac{T_{AB}c}{J_{AB}} \qquad 8000\,psi = \frac{T_0(0.375\,\text{in.})}{\frac{\pi}{2}(0.375\,\text{in.})^4}$$

$$T_0 = 663\,\text{lb}\cdot\text{in.}$$

$$\tau_{max} = \frac{T_{CD}c}{J_{CD}} \qquad 8000\,psi = \frac{2.8T_0(0.5\,\text{in.})}{\frac{\pi}{2}(0.5\,\text{in.})^4}$$

$$T_0 = 561\,\text{lb}\cdot\text{in.} \qquad \boxed{T_0 = 561\,\text{lb}\cdot\text{in}}$$

- Find the corresponding angle of twist for each shaft and the net angular rotation of end A



$$\phi_{A/B} = \frac{T_{AB}L}{J_{AB}G} = \frac{(561\,\text{lb}\cdot\text{in.})(24\,in.)}{\frac{\pi}{2}(0.375\,\text{in.})^4\left(11.2\times10^6\,\text{psi}\right)}$$

$$= 0.387\,\text{rad} = 2.22^o$$

$$\phi_{C/D} = \frac{T_{CD}L}{J_{CD}G} = \frac{2.8(561\,\text{lb}\cdot\text{in.})(24\,in.)}{\frac{\pi}{2}(0.5\,\text{in.})^4\left(11.2\times10^6\,\text{psi}\right)}$$

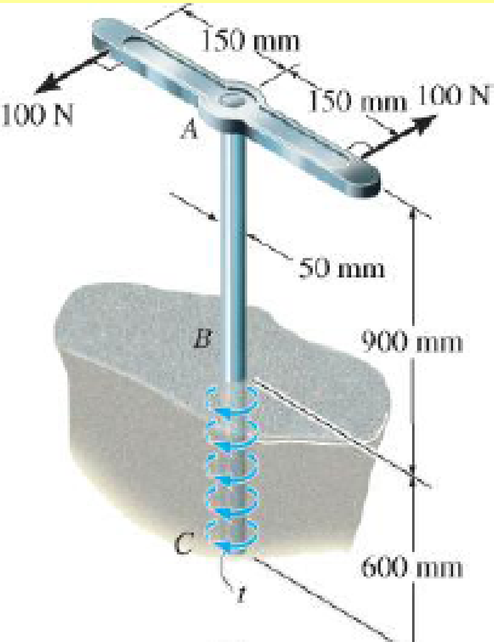
$$= 0.514\,\text{rad} = 2.95^o$$

$$\phi_B = 2.8\phi_C = 2.8\left(2.95^o\right) = 8.26^o$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.26^o + 2.22^o \qquad \boxed{\phi_A = 10.48^o}$$

Example

50-mm-diameter solid cast-iron post shown is buried 600 mm in soil. Determine maximum shear stress in the post and angle of twist at its top. Assume torque about to turn the post, and soil exerts uniform torsional resistance of t N·mm/mm along its 600 mm buried length. $G = 40(10^3)$ GPa



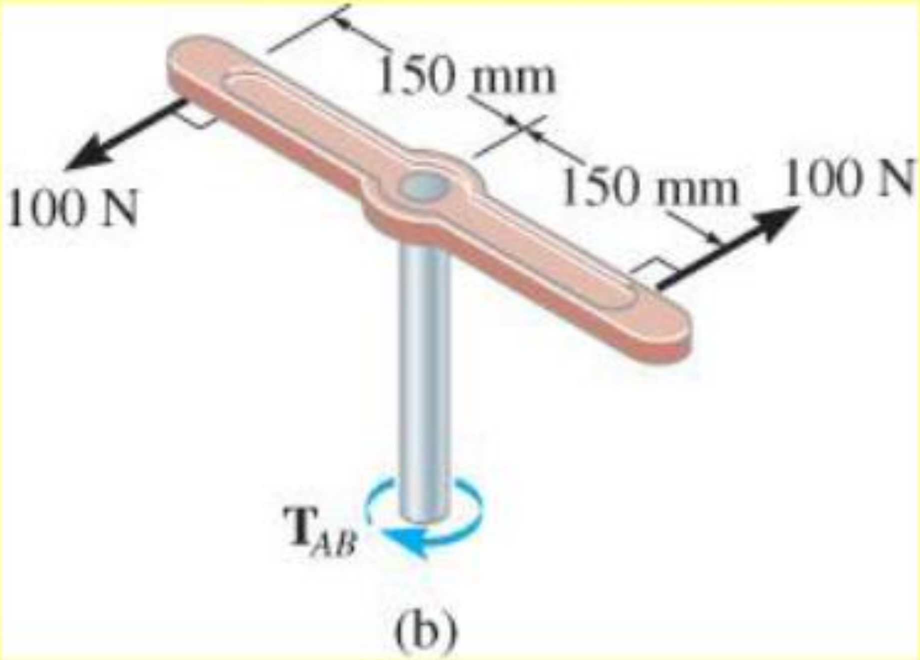
(a)

Example

Internal torque

From free-body diagram

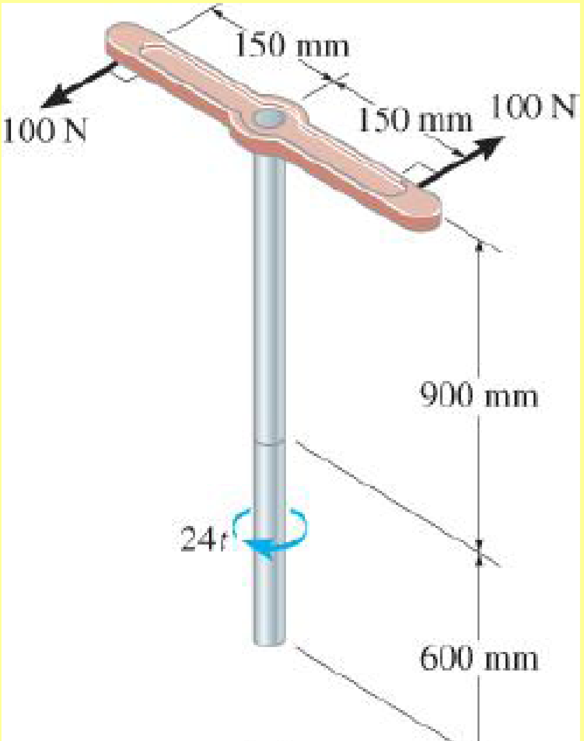
$$\Sigma M_z = 0; \qquad T_{AB} = 100\ \text{N}(300\ \text{mm}) = 30 \times 10^3\ \text{N·mm}$$



Example

Internal torque

Magnitude of the uniform distribution of torque along buried segment BC can be determined from equilibrium of the entire post.



(c)

$$\Sigma M_z = 0;$$

$$100\ \text{N}(300\ \text{mm}) - t(600\ \text{mm}) = 0$$

$$t = 50\ \text{N·mm}$$

Example

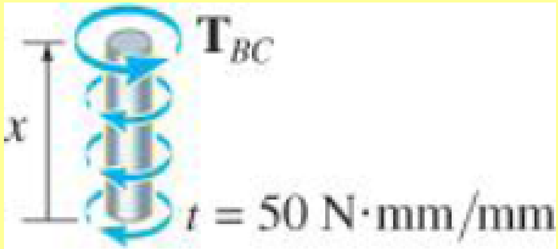
Internal torque

Hence, from free-body diagram of a section of the post located at position x within region BC, we have

$$\Sigma M_z = 0;$$

$$T_{BC} - 50x = 0$$

$$T_{BC} = 50x$$



(d)

Example

Maximum shear stress

Largest shear stress occurs in region AB, since torque largest there and J is constant for the post. Applying torsion formula

$$\tau_{max} = \frac{T_{AB}\,c}{J} = \ldots = 1.22\ \mathrm{N/mm^2}$$

Example

Angle of twist

Angle of twist at the top can be determined relative to the bottom of the post, since it is fixed and yet is about to turn. Both segments AB and BC twist, so

$$\phi_A = \frac{T_{AB} L_{AB}}{JG} + \int_0^{L_{BC}} \frac{T_{BC}\, dx}{JG}$$

$$\vdots$$

$$\phi_A = 0.00147 \text{ rad}$$

Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
  - power
  - speed

- Designer must select shaft material and cross-section to meet performance specifications without exceeding allowable shearing stress.

- Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi f T$$

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

- Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2} c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

Power transmitting

- ***Power*** (***P***) is defined as work performed per unit of time:

$$P = T\omega$$

- Since shaft's angular velocity

$$\omega = 2\pi N/60$$

Then we can express power as:

$$P = T\,(2\pi N/60) = T\,(2\pi f)$$

- *Frequency f :* It measures the number of cycles per second and 1 cycle = 2 radians,
- *N* is a shaft's rotation in ***rpm*** together with the power of the motor **in hp or kw are** often reported.

Power transmitting

$$T = 5252 \frac{\text{Power }_{(\text{hp})}}{N_{rpm}} \Rightarrow\Rightarrow (lb.ft)$$

$$T = 9549 \frac{\text{Power }_{(\text{Kw})}}{N_{rpm}} \Rightarrow\Rightarrow (N.m)$$

$$\text{Power }_{(\text{hp})} = \frac{T_{(lb.ft)} . N_{rpm}}{5252} \Rightarrow\Rightarrow hp \quad \text{or}$$

$$\text{Power }_{(\text{kw})} = \frac{T_{(N.m)} . N_{rpm}}{9549} \Rightarrow\Rightarrow kw$$

Where: 1hp = 0.746kw and 1 lb.ft = 1.356 N.m

Power transmitting

- Principal transmission shaft performance specifications are:
  - power : *hp or kw*
  - speed : *N in rpm*

- Designer must select shaft material and cross-section to meet performance specifications without exceeding **allowable shearing stress (τ_{all} or τ_{max}).**

- **<u>Determine torque</u> applied to shaft at <u>specified power (P)</u> and speed (N),**

$$T = 5252\frac{\text{Power}_{\text{(hp)}}}{N_{rpm}} \Rightarrow\Rightarrow (lb.ft)$$

$$T = 9549\frac{\text{Power}_{\text{(Kw)}}}{N_{rpm}} \Rightarrow\Rightarrow (N.m)$$

- **Find shaft diameter (*d*) which will not exceed the maximum allowable shearing stress,**

$$\tau_{\max} = \frac{Tc}{J}$$

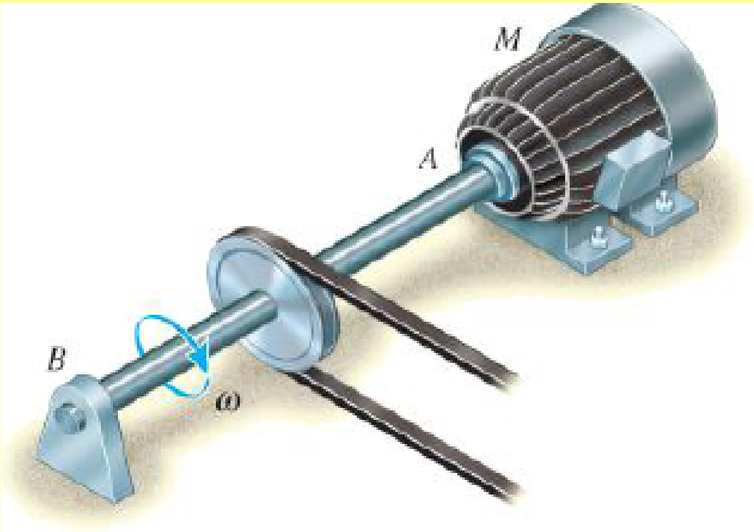
$$\frac{J}{c} = \frac{\pi}{2}c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

Example

Solid steel shaft shown used to transmit 3750 W from attached motor **M**. Shaft rotates at $N = 175$ rpm and the steel $\tau_{\text{allow}} = 100$ MPa. <u>**Determine**</u> required diameter of shaft (d) to nearest mm.

$$\tau_{all} = \tau_{\max} = \frac{T\,r}{J}$$



$$\frac{J}{r} = \frac{T}{\tau_{\text{allow}}}$$

$$r^3 = \frac{2}{\pi}\frac{T}{\tau_{allow}}$$

$J = (\pi/2)r^4$

Example

Given: $P = 3750\ w = 3.75\ kw,\ N = 175\ rpm,\ and$

$\tau_{all} = 100\ MPa = 100x10^6 N/m^2$

$$T = 9549 \frac{\text{Power (Kw)}}{N_{rpm}} \Rightarrow\Rightarrow_{(N.m)}$$

Thus, $\quad T = 204.6\ N \cdot m$

$$r^3 = \frac{2}{\pi}\frac{T}{\tau_{allow}} = \frac{2x204.6}{\pi c100x10^6} = 1.303x10^{-6} m^3$$

$$r = 0.0109m = 10.9mm \approx 11mm$$
$$d = 2r = 22mm$$

We select shaft with diameter of $d = 22$ mm

Example

Torque on shaft determined from $P = T\omega$,

Thus, $P = 3750\ \mathrm{N \cdot m/s}$

$$\omega = \frac{175\ \mathrm{rev}}{\mathrm{min}}\left(\frac{2\pi\ \mathrm{rad}}{1\ \mathrm{rev}}\right)\left(\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right) = 18.33\ \mathrm{rad/s}$$

Thus, $P = T\omega$, $T = 204.6\ \mathrm{N \cdot m}$

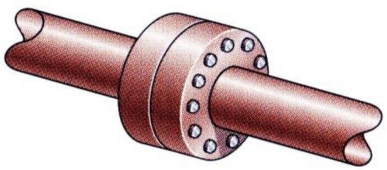
$$\frac{J}{c} = \frac{\pi c^4}{2\,c^2} = \frac{T}{\tau_{\mathrm{allow}}}$$

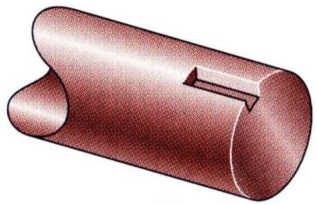
$$\vdots$$

$$c = 10.92\ \mathrm{mm}$$

Since $2c = 21.84\ \mathrm{mm}$, select shaft with diameter of $d = 22\ \mathrm{mm}$

Stress Concentrations





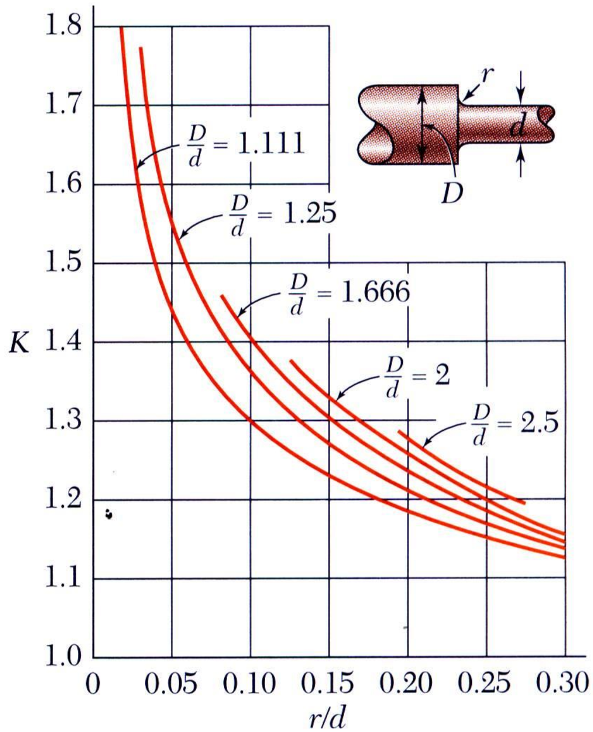


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

  assumed a circular shaft with uniform cross-section loaded through rigid end plates.

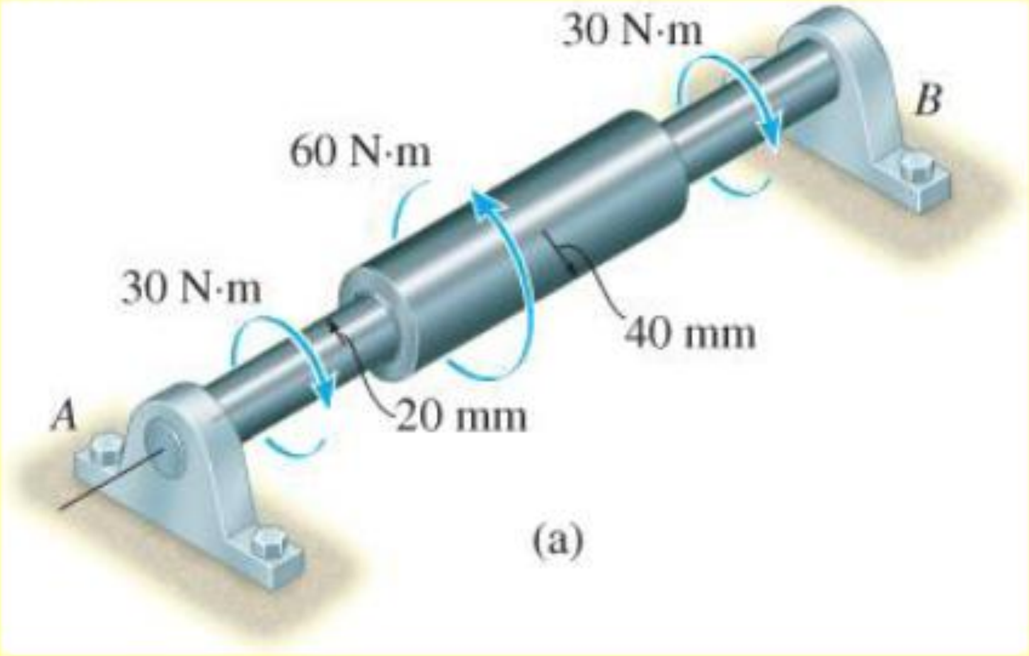
- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$

Example

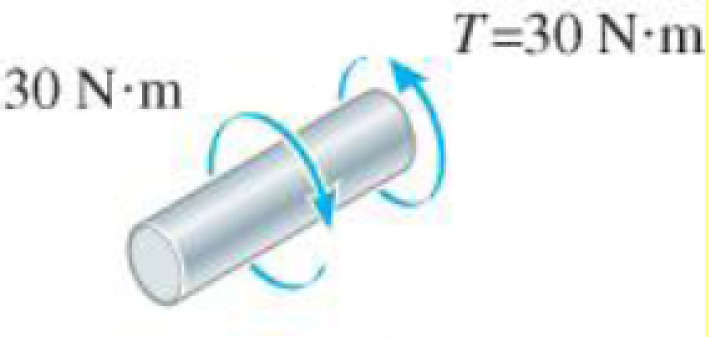
Stepped shaft shown is supported at bearings at A and B. Determine maximum stress in the shaft due to applied torques. Fillet at junction of each shaft has radius $r = 6$ mm.



Example

Internal torque

By inspection, moment equilibrium about axis of shaft is satisfied. Since maximum shear stress occurs at rooted ends of smaller diameter shafts, internal torque $(30\ \mathrm{N{\cdot}m})$ can be found by applying method of sections



(b)

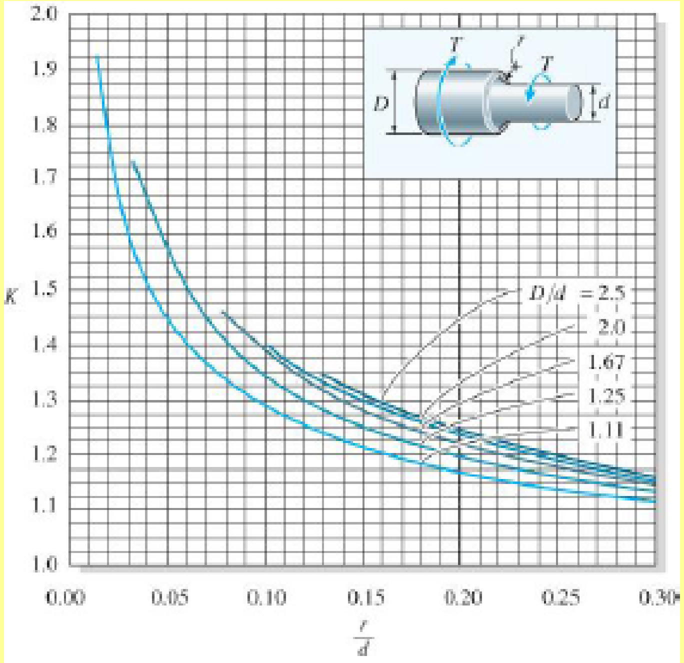
Example

Maximum shear stress

From shaft geometry, we have

$$\frac{D}{d} = \frac{2(40\text{ mm})}{2(20\text{ mm})} = 2$$

$$\frac{r}{d} = \frac{6\text{ mm})}{2(20\text{ mm})} = 0.15$$



Thus, from the graph, $K = 1.3$

$$\tau_{\max} = K(Tc/J) = \ldots = 3.10\text{ MPa}$$

Example

Maximum shear stress

From experimental evidence, actual stress distribution along radial line of x-section at critical section looks similar to: